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QUASI-ALTERNATING LINKS AND POLYNOMIAL INVARIANTS

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ABSTRACT. In this note, we survey several criteria for knots and links to be quasi-alternating by using polynomial invariants such as Q-polynomials and Kauffman polynomials. Also, we mention two new generalizations of quasi-alternating links.

1. INTRODUCTION

Alternating knots and links give a classical but remarkable class of knots and links. The definition is described through diagrams, but it is very recent that a characterization without involving diagrams was found by Greene [8] and Howie [12] independently.

On the other hand, there are a lot of generalizations of alternating knots and links in knot theory. Here is a list of adjectives, which is not complete.

- almost alternating, m-almost alternating (Adams et al. [1])
- toroidally alternating (Adams [2])
- adequate (Lickorish-Thistlethwaite [17])
- semi-alternating (Lickorish-Thistlethwaite [17])
- alternative (Kauffman [14])
- pseudo-alternating (Mayland-Murasugi [19])
- n-semi-alternating (Beltrami [3])
- algebraically alternating (Ozawa [20])
- quasi-alternating (Ozsváth-Szabó [21])

The objects of this note are quasi-alternating knots and links introduced by Ozsváth and Szabó in their Heegaard Floer homology theory.

Quasi-alternating links (abbreviated as QA links) are defined recursively as follows.

(1) The unknot is QA.

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(2) If a link L has a diagram with QA-crossing, then L is QA.
Here, a QA-crossing is a crossing where two resolutions L_∞, L_0 as illustrated in Figure 1 satisfy that
(a) both L_∞ and L_0 are QA, and
(b) $\det L = \det L_\infty + \det L_0$.

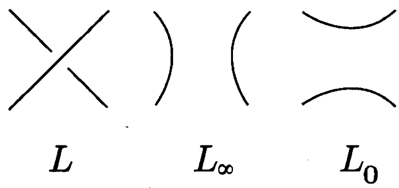


FIGURE 1. Two resolutions L_∞ and L_0

For a link L, its determinant $\det L$ is a non-negative integer. We should remark that if a link L is QA, then $\det L > 0$. Also, Ozsváth-Szabó [21] showed that any alternating knot and non-split alternating link are QA.

Because of its recursive definition, it is not easy to identify whether a given knot or link is QA or not.

Problem 1.1. *Decide whether a given knot or link is QA or not.*

Example 1.2. The knot 8_{21} is non-alternating, but QA. As illustrated in Figure 2, the marked crossing in the first diagram is a QA-crossing. For, each of two resolutions is alternating, so QA, and we have the desired equality among their determinants.

There are several properties of QA links:

- The double branched cover is an L-space.
- The double branched cover bounds a negative-definite 4-manifold W with $H_1(W) = 0$.
- Homologically thin (knot Floer, reduced Khovanov, and reduced odd Khovanov homologies are thin, i.e. supported on a single diagonal.)

Here is a digression. Let K be the (-2)-twist knot, which is the knot 5_2 in the knot table. See Figure 3.

Since K is 2-bridge, its double branched cover is a lens space, which is a typical L-space as its name suggests. Then, how about the 3-fold cyclic branched cover? A direct approach is to calculate its Heegaard Floer homology. As far as we know, there are some references [9, 16] concerning Heegaard Floer homology of cyclic branched covers. Although we do not deny this approach, it would be hard to execute.

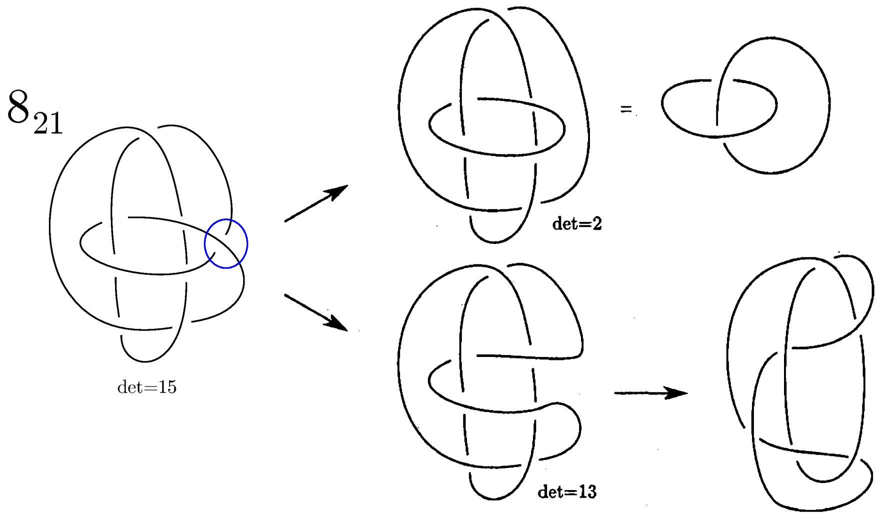


FIGURE 2. The knot 8_{21} is QA.

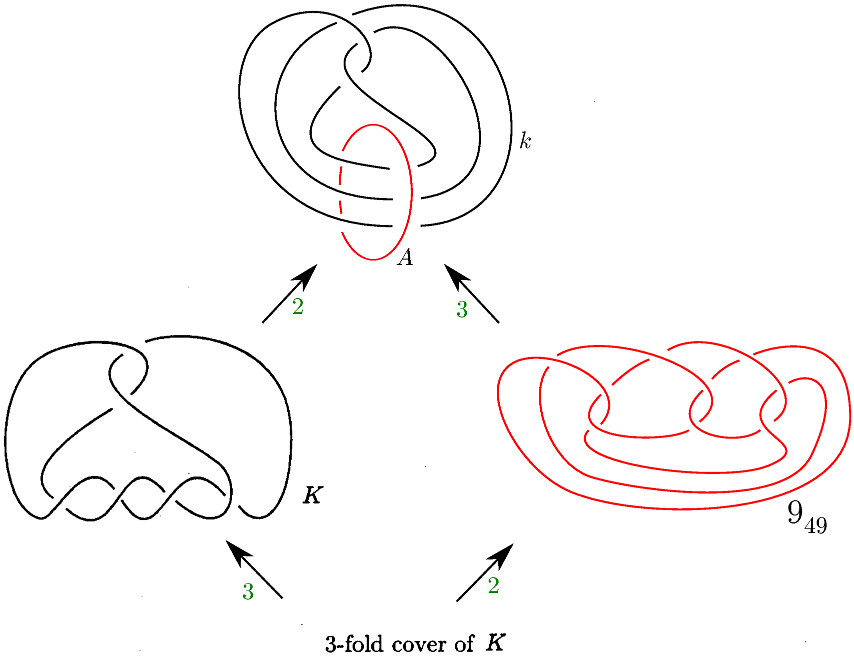


FIGURE 3. The 3-fold cyclic branched cover of the knot 5_2 is an L-space.

However, there is a detour. Since K is 2-bridge, it admits a cyclic period of order two. The image of K under this cyclic action is denoted by k in Figure 3. There, A is the image of the axis. We can see that the factor knot k is unknotted. Hence the 3-fold cyclic branched cover of k remains to be the 3-sphere, and the lift of A gives the knot 9_{49}. Thus, the 3-fold cyclic branched cover of the original knot K is homeomorphic to the double branched cover of 9_{49}. In fact, 9_{49} is QA, so its double branched cover is an L-space. By the same technique, the 4- and 5-fold cyclic branched covers of K are shown to be L-spaces without any calculation of Heegaard Floer homology [26, 11].

2. Criteria by Q-polynomial

As mentioned before, it is not easy to determine whether a given knot or link is QA or not, in general. However, Qazaqzeh and Chbili [22] found a very simple criterion for QA links in terms of Q-polynomials.

Theorem 2.1 ([22]). *If a link L is QA, then*

$$\deg Q_L \leq \det L - 1,$$

where $\deg Q_L$ *is the maximal degree of the Q-polynomial Q_L of L.*

We recall the definition of Q-polynomials [4, 10]. Let L be an unoriented link. Then its Q-polynomial $Q_L(x)$ is a Laurent polynomial satisfying the following.

(1) $Q_U = 1$, where U is the unknot.
(2) $Q_{L_+} + Q_{L_-} = x(Q_{L_\infty} + Q_{L_0})$ holds for the skein quadruple $(L_+, L_-, L_\infty, L_0)$ as illustrated in Figure 4.

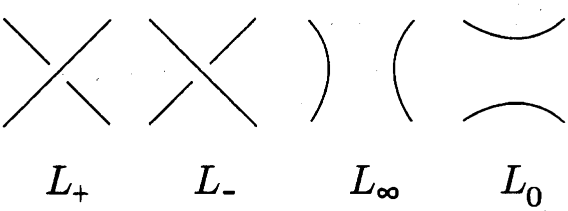


FIGURE 4. The skein quadruple

For knots, their Q-polynomials have no negative powers of x.

Example 2.2. Let K be the knot 8_{19}, which is non-alternating. In fact, K is the $(3,4)$-torus knot. Then $\deg Q_K = 7$ and $\det K = 3$. Hence K is not QA by Theorem 2.1.

The key of the argument of Qazaqzeh and Chbili [22] is the next observation.

Lemma 2.3. *Let L be a link, and let L_0 and L_∞ be two resolutions at some crossing of a diagram of L. Then*

$$\deg Q_L \leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1.$$

Proof of Theorem 2.1. It is an induction on determinant. Let L be a QA link. If $\det L = 1$, then L is the unknot. Hence $Q_L = 1$, so the inequality $\deg Q_L \leq \det L - 1$ holds.

Suppose $\det L > 1$. Let L_0 and L_∞ be two resolutions at a QA-crossing of L. Thus these are QA, and $\det L_* < \det L$ for $* \in \{0, \infty\}$. By Lemma 2.3,

$$\begin{aligned}\deg Q_L &\leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1\\ &< \max\{\det L_0, \det L_\infty\} + 1\\ &\leq \det L_0 + \det L_\infty = \det L.\end{aligned}$$

□

In [24], we gave an improvement of the criterion (Theorem 2.1) of Qazaqzeh and Chbili.

Theorem 2.4 ([24]). *If a link L is QA, then one of the following holds.*

(1) *L is a $(2, n)$-torus link ($n \neq 0$) and $\deg Q_L = \det L - 1$; or*
(2) $\deg Q_L \leq \det L - 2$.

Example 2.5. Here are two examples which show that the evaluation of Theorem 2.4(2) is optimal.

(1) Let K be the figure-eight knot. It is alternating, so QA, and $\deg Q_K = 3$, $\det K = 5$.
(2) Let L be the connected sum of two Hopf links. Since L is non-split alternating, it is QA. And $\deg Q_L = 2$, $\det L = 4$.

Example 2.6. Each of non-alternating knots 12_{n0025}, 12_{n0093}, 12_{n0115}, 12_{n0138}, 12_{n0199}, 12_{n0355}, 12_{n0374} has $\deg Q = 10, \det = 11$. None of these is QA by our criterion (Theorem 2.4). This cannot be deduced by Theorem 2.1.

Here is a brief sketch of the proof of Theorem 2.4. The proof uses an induction on determinant. Let L be a non-trivial QA link. Then the resolution at a QA crossing gives two QA links L_∞ and L_0. The argument is split into three cases.

(1) Neither L_∞ nor L_0 is a $(2, n)$-torus link. By the inductive hypothesis, $\deg Q_{L_*} \leq \det L_* - 2$ for $* \in \{\infty, 0\}$. Then,

$$\begin{aligned}\deg Q_L &\leq \max\{\deg Q_{L_\infty}, \deg Q_{L_0}\} + 1\\ &= \deg Q_{L_\alpha} + 1 \qquad (\{\alpha, \beta\} = \{\infty, 0\})\\ &\leq (\det L_\alpha - 2) + 1\\ &= (\det L - \det L_\beta) - 1\\ &\leq \det L - 2.\end{aligned}$$

(2) The case where one of L_∞, L_0 is a $(2,n)$-torus link is also easy.
(3) If both are $(2,*)$-torus links, then we need another argument involving Dehn surgery. See [24].

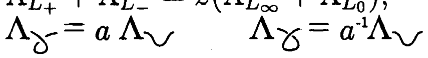
3. Criteria by Kauffman polynomial

The previous argument in Section 2 works for Kauffman polynomial, which is a two-variable generalization of Q-polynomial [13].

Theorem 3.1. *For a QA link L, either*

(1) *L is a $(2,n)$-torus link* $(n \neq 0)$, *and* $\deg_z F_L = \det L - 1$; *or*
(2) $\deg_z F_L \leq \det L - 2$.

For a diagram D of an oriented link L, $\Lambda_D(a,z)$ is defined with forgetting its orientation as follows:

(1) Λ_D is a regular isotopy invariant;
(2) For the unknot diagram without crossing U, $\Lambda_U = 1$;
(3) $\Lambda_{L_+} + \Lambda_{L_-} = z(\Lambda_{L_\infty} + \Lambda_{L_0})$;
(4) 

If D has writhe w, then the Kauffman polynomial of L is defined as

$$F_L(a,z) = a^{-w}\Lambda_D(a,z).$$

Since $F_L(1,z) = Q_L(z)$, we have $\deg Q_L \leq \deg_z F_L$, where $\deg_z F_L$ is the maximal degree of variable z.

For alternating ones among QA links, a classical fact by R. Crowell [6] implies the following.

Theorem 3.2. *For a non-split alternating link L, either*

(1) *L is a $(2,n)$-torus link* $(n \neq 0)$, *and* $\deg_z F_L = \det L - 1$;
(2) *L is the figure-eight knot or Hopf link # Hopf link, and* $\deg_z F_L = \det L - 2$; *or*
(3) $\deg_z F_L \leq \det L - 3$.

For non-alternating QA links, we have the following.

Theorem 3.3 ([25]). *For non-alternating QA link L, either*

(1) $\deg_z F_L \leq \det L - 3$; *or*
(2) *L has exactly 3 components, each of which is unknotted. Moreover, L is obtained from the Hopf link by a banding on one component.*

We expect that the second possibility of Theorem 3.3 would not happen, but we could not erase it. As an immediate corollary of Theorem 3.3, we have the following criterion for non-alternating QA knots.

Corollary 3.4. *For a non-alternating QA knot K, we have*

$$\deg Q_K \leq \deg_z F_K \leq \det K - 3.$$

Example 3.5. The evaluation of Corollary 3.4 is sharp. Let K be the $(-3, 2, n)$-pretzel knot, $n \geq 3$ odd. This knot has the following properties.

- K is non-alternating QA.
- $\det K = n + 6$.
- $\deg Q_K = \deg_z F_K = n + 3$.

Example 3.6. Let $K = 9_{46}$, which is the $(-3, 3, 3)$-pretzel knot. Then it satisfies:

- K is non-alternating.
- $\det K = 9$.
- $\deg Q_K = \deg_z F_K = 7$.

Hence, K is not QA by Corollary 3.4. This fact was known by its thick Khovanov homology (see [5, page 2456]).

Finally, we propose a problem on the a-span, denoted by $\mathrm{span}_a F_L$, of the Kauffman polynomial $F_L(a, z)$ for QA link L. If L is non-split alternating, then $\mathrm{span}_a F_L$ is equal to its crossing number by [27]. Hence the inequality $\mathrm{span}_a F_L \leq \det L$ holds. We expect that this would hold for QA links.

Problem 3.7. *Let L be a QA link.*

(1) *Show that* $\mathrm{span}_a F_L \leq \det L$.
(2) *Show that* $\mathrm{span}_a F_L \leq \mathrm{span}\, V_L \leq \det L$, *where V_L is the Jones polynomial of L.*

These are verified for all QA knots up to 11 crossings. The second inequality $\mathrm{span} V_L \leq \det L$ of Problem 3.7(2) is mentioned in [22].

4. Q-POLYNOMIAL VERSUS KAUFFMAN POLYNOMIAL

It is possible that $\deg Q_L < \deg_z F_L$. Hence there is a chance that the criterion (Theorem 3.3) by the Kauffman polynomial is strictly stronger than one (Theorem 2.4) by the Q-polynomial. The next shows that it can happen.

Theorem 4.1. *There exist infinitely many hyperbolic knots and links L_n such that*

(1) *L_n is not QA;*
(2) $\deg Q_{L_n} = \det L_n - 4$; *and*
(3) $\deg_z F_{L_n} = \det L_n$.

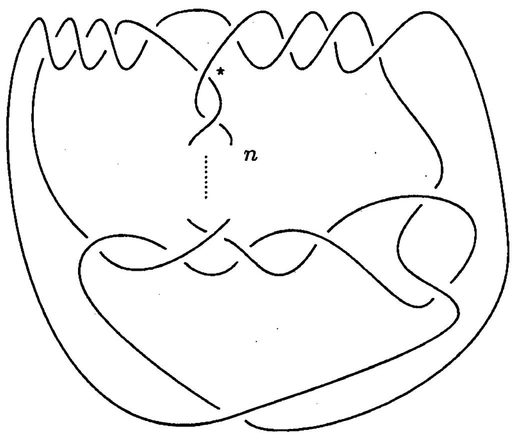


FIGURE 5. The link L_n

In fact, it can be shown ([25]):

- L_n is a knot if n is odd, has two components if n is even.
- $\det L_n = n + 10$.
- $\deg Q_{L_n} = n + 6$ $(n \geq 3)$.
- $\deg_z F_{L_n} = n + 10$ $(n \geq 1)$.

Thus L_n is detected to be non-QA by Theorem 3.3, but not by Theorem 2.4.

5. QA LINKS WITH SMALL DETERMINANT

Greene [7] conjectures that there are only finitely many QA links with a given determinant. He determined all QA knots and links with determinant ≤ 3 as shown in Table 1.

det	quasi-alternating knot/link
1	unknot
2	Hopf link
3	trefoil

TABLE 1. QA links with determinant ≤ 3

We proved in [24, 25] the followings.

Theorem 5.1. *If L is a QA link with* $\det L = 4$, *then L is the* $(2, \pm 4)$-*torus link, or L has 3 components, each of which is unknotted, and* $\deg_z F_L \leq 2$.

Theorem 5.2. *If L is a QA link with* $\det L = 5$, *then L is either the figure-eight knot or the* $(2, \pm 5)$-*torus knot.*

After that, Lidman and Sivek [18] classified all QA links with $\det \leq 7$ based on the determination of all formal L-spaces with order at most 7.

Theorem 5.3 ([18]). *QA links with* $\det \leq 7$ *are* 2*-bridge or a connected sum of* 2*-bridge links.*

Thus all QA links with $\det \leq 7$ are determined as in Table 2.

det	quasi-alternating knot/link
1	unknot
2	Hopf link
3	trefoil
4	$(2, \pm 4)$-torus link, Hopf $\sharp$ Hopf
5	$(2, \pm 5)$-torus knot, figure-eight knot
6	$(2, \pm 6)$-torus link, trefoil $\sharp$ Hopf link
7	$(2, \pm 7)$-torus knot, 5_2

TABLE 2. QA links with determinant ≤ 7

Problem 5.4. (1) *Solve Greene's conjecture.*
(2) *Determine QA links with* $\det = 8$.

We remark that the pretzel link $P(-3, 2, 2)$ is non-alternating QA and $\det = 8$.

6. WEAKLY QUASI-ALTERNATING LINKS

In the remaining two sections, we mention two recent generalizations of QA links. The first one is weakly quasi-alternating links introduced by D. Kriz and I. Kriz [15].

Weakly quasi-alternating links (abbreviated as WQA links) are defined recursively as follows.

(1) The unknot and unlinks are WQA.
(2) If a link L has a diagram with WQA-crossing, then L is WQA. Here, a WQA-crossing is a crossing where two resolutions L_∞, L_0 satisfy
   (a) both L_∞ and L_0 are WQA, and
   (b) $\det L = \det L_\infty + \det L_0$.

For a split link, its determinant is 0. Hence, any split link is WQA. Thus we think that this class would be too wide.

Kriz-Kriz [15] showed:

Theorem 6.1 ([15]). (1) *Any WQA link is BOS thin.*

(2) *The double branched cover of a WQA knot is an L-space.*

Baldwin-Ozsváth-Szabó cohomology H_{BOS} is an invariant of oriented links. A link L is BOS thin if

$$\operatorname{rank} H^i_{BOS}(L) = \begin{cases} \det L, & \text{if } i = \sigma(L)/2, \\ 0, & \text{otherwise.} \end{cases}$$

For QA links, Greene conjectures that there are only finitely many QA links with a given determinant, but the same thing does not hold for WQA links.

Theorem 6.2. *Let $d \geq 0$ be a multiple of 4 or a square (> 1). Then there exist infinitely many WQA, non-QA links with* $\det = d$.

Example 6.3. The $(-2, 2, n)$-pretzel link P_n has $\det = 4$ for any integer n. For example, P_0 is Hopf link # Hopf link, P_1 is the $(2, 4)$-torus link. Also, $\deg Q_{P_n} = |n| + 2$. Hence P_n is not QA if $|n| \geq 2$, but P_n is WQA as illustrated in Figure 6.

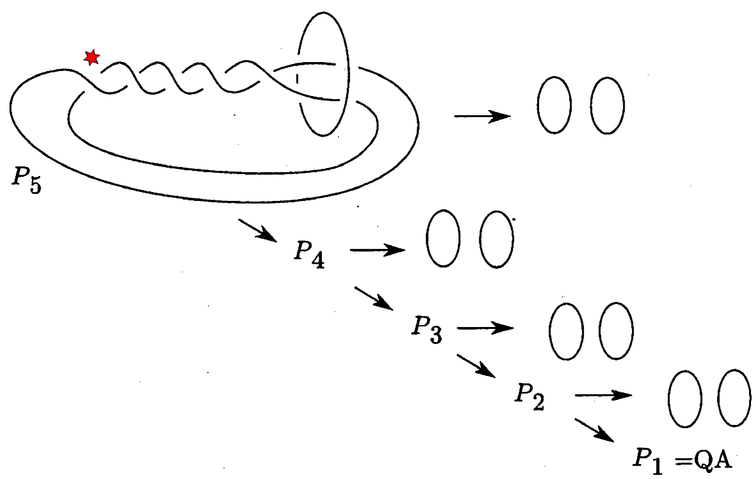


FIGURE 6. WQA links P_n with $\det = 4$

Although we do not give the proof of Theorem 6.2, the pretzel link $P(-l, l, m)$ $(3 \leq l \leq m)$ gives an example for a square determinant. Let $L = P(-l, l, m)$. Then $\det L = l^2$, and any crossing in the m-twist strand is WQA. By [7], L is not QA.

Also, any Kanenobu knot is shown to be WQA. They have determinant 25, and it is known that there are only finitely many QA Kanenobu knots [22].

Question 6.4. *Let $1 \leq d \leq 3$. Is there a WQA, non-QA link with* $\det = d$*?*

7. Two-fold quasi-alternating links

Scaduto and Stoffregen [23] introduced two-fold quasi-alternating links. We will not give full details (see [23]). For a link, a marking w assigns 0 or 1 to each component of L. The weight 1 is expressed as one dot on the component. The total number of dots is required to be even. After a resolution, the dots are carried in the natural way.

Two-fold quasi-alternating links (abbreviated as TQA links) are defined recursively as follows.

(1) The unknot with trivial marking is TQA.
(2) A split union of two odd-marked links is TQA.
(3) L is TQA if it has TQA crossing where two resolutions L_∞ and L_0 satisfy
   (a) both of L_∞ and L_0 are TQA,
   (b) $\det L = \det L_\infty + \det L_0$.

It is not hard to see that QA $\Longrightarrow$ TQA $\Longrightarrow$ WQA, in general. As a typical example, Figure 7 shows that the non-QA knot 11_{n50} is TQA. (Dots on the same component is counted mod 2.)

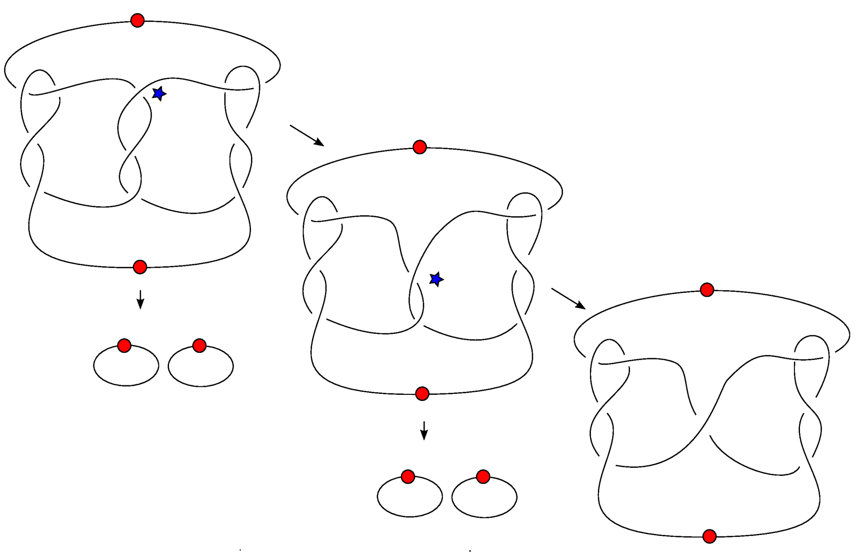


Figure 7. The non-QA knot 11_{n50} is TQA.

It is shown in [23] that a TQA link is mod 2 Khovanov thin. Also, the framed instanton homology of the double branched cover of a TQA link is examined there.

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